

On the Ambarzumyan's theorem for the Quasi-periodic Problem

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Abstract

We obtain the classical Ambarzumyan's theorem for the Sturm-Liouville operators $L_t(q)$ with $q \in L^1[0, 1]$ and quasi-periodic boundary conditions, $t \in [0, 2\pi)$, when there is not any additional condition on the potential q .

Keywords: Ambarzumyan theorem; inverse spectral theory; Hill operator

1. Introduction

In this study we consider the Sturm-Liouville operator $L_t(q)$ generated in the space $L^2[0, 1]$ by the expression

$$-y'' + q(x)y \tag{1}$$

and the quasi-periodic boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0), \tag{2}$$

where $q \in L^1[0, 1]$ is a real-valued function and t is a fixed real number in $[0, 2\pi)$. Note that the operator $L_t(q)$ is self-adjoint and the cases $t = 0$ and $t = \pi$ correspond to the periodic and antiperiodic problems, respectively. Since the spectrum $S(L(q))$ of Hill operator $L(q)$ generated in the space $L^2(-\infty, \infty)$ by expression (1) with periodic potential q is the union of the spectra $S(L_t(q))$ of the operators $L_t(q)$ for $t \in [0, 2\pi)$ (e.g., see [1]), the operators $L_t(q)$ have a fundamental role in the spectral theory of the operator

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$L(q)$. In 1929, Ambarzumyan [2] obtained the following theorem considered as the first theorem in inverse spectral theory:

If $\{n^2 : n = 0, 1, \dots\}$ is the spectrum of the Sturm-Liouville operator with Neumann boundary condition, then $q = 0$ a.e.

In [3], Chern and Shen proved Ambarzumyan's theorem for the Sturm-Liouville differential systems with Neumann boundary conditions. Later, in [4], by imposing an additional condition on the potential they extended the classical Ambarzumyan's theorem for the Sturm-Liouville equation to the general separated boundary conditions. See basics and further references in [5, 6].

At this point we refer in particular to [7, 8]. In [7], for the vectorial Sturm-Liouville problem under periodic or antiperiodic boundary conditions, Yang-Huang-Yang found two analogs of Ambarzumyan's theorem. Their result supplements the Pöschel-Trubowitz inverse spectral theory [9]. More recently, Cheng-Wang-Wu [8] proved the following theorem:

(a) *If all eigenvalues of the operator $L_0(q)$ are nonnegative and they include $\{(2n\pi)^2 : n \in \mathbb{N}\}$, then $q = 0$ a.e.*

(b) *If all eigenvalues of the operator $L_\pi(q)$ are not less than π^2 and they include $\{(2n\pi - \pi)^2 : n \in \mathbb{N}\}$, and*

$$\int_0^1 q(x) \cos(2\pi x) dx \geq 0, \quad (3)$$

then $q = 0$ a.e.

The present work was stimulated by the papers [4, 8]. For the first time, we obtain Ambarzumyan's theorem for the operator $L_t(q)$ with $t \in [0, 2\pi)$, generated by quasi-periodic boundary conditions (2). The result established below show that the potential q can be determined from one spectrum and there is not any additional condition on q such as (3) for the operator $L_t(q)$ with $t = \pi$ (see also [4, 7]). The result of this paper is the following.

Theorem 1. *If first eigenvalue of the operator $L_t(q)$ for any fixed number t in $[0, 2\pi)$ is not less than the value of $\min\{t^2, (2\pi - t)^2\}$ and the spectrum $S(L_t(q))$ contains the set $\{(2n\pi - t)^2 : n \in \mathbb{N}\}$, then $q = 0$ a.e.*

2. Preliminaries and Proof of the result

We now introduce some preliminary facts. In [10] (see also [?]), without using the assumption $q_0 = 0$, they proved the following result:

The eigenvalues $\lambda_n(t)$ of the operator $L_t(q)$ for $q \in L^1[0, 1]$ and $t \neq 0, \pi$, satisfy the following asymptotic formula

$$\lambda_n(t) = (2\pi n + t)^2 + q_0 + O(n^{-1} \ln|n|) \quad \text{as } |n| \rightarrow \infty, \quad (4)$$

where $q_n = (q, e^{i2\pi nx})$ for $n \in \mathbb{Z}$ and (\cdot, \cdot) is the inner product in $L^2[0, 1]$.

Note that when $q = 0$, $(2\pi n + t)^2$ for $n \in \mathbb{Z}$ is the eigenvalue of the operator $L_t(0)$ for any fixed $t \in [0, 2\pi)$ corresponding to the eigenfunction $e^{i(2\pi n + t)x}$.

PROOF OF THEOREM 1. Using the assumption that, for any $n \in \mathbb{N}$, $(2n\pi - t)^2$ belongs to the spectrum $S(L_t(q))$ and taking into account that, for sufficiently large $|n|$, the asymptotic formulas (4) for $t \neq 0, \pi$, and, in [8], (1.2)-(1.3) for $t = 0, \pi$ (see Theorem 1.1. of [8]), we obtain

$$q_0 = \int_0^1 q(x) dx = 0. \quad (5)$$

Let us show that, for fixed $t \in [0, 2\pi)$, the first eigenvalue of the operator $L_t(q)$ is either t^2 or $(2\pi - t)^2$ corresponding to the eigenfunctions $y = e^{itx}$ or $y = e^{i(-2\pi + t)x}$, respectively. First, suppose that the value of $\min\{t^2, (2\pi - t)^2\}$ is t^2 . By the variational principle and (5), we have for $y = e^{itx}$

$$t^2 \leq \lambda_0(t) \leq \frac{\int_0^1 -\bar{y}y'' dx + \int_0^1 q(x)|y|^2 dx}{(y, y)} = t^2 + q_0 = t^2. \quad (6)$$

This implies that the first eigenvalue of the operator $L_t(q)$ is $\lambda_0(t) = t^2$ and the test function $y = e^{itx}$ is the first eigenfunction of the operator. Thus, Substituting the expressions $y = e^{itx}$ and $\lambda_0(t) = t^2$ into the equation

$$-y'' + q(x)y = \lambda y,$$

we get $q = 0$ in $L^1[0, 1]$. Similarly, one can readily show that if the value of $\min\{t^2, (2\pi - t)^2\}$ is $(2\pi - t)^2$, then the function $y = e^{i(-2\pi + t)x}$ is the first eigenfunction corresponding to the first eigenvalue $(2\pi - t)^2$ and $q = 0$ in $L^1[0, 1]$. \square

Remark 1. Note that instead of the subset $\{(2n\pi - t)^2 : n \in \mathbb{N}\}$ of the spectrum $S(L_t(q))$ in Theorem 1 if we use either of the subsets

$\{(2n\pi + t)^2 : n \in \mathbb{N}\}$, $\{m^2 : m \text{ is either } (2n\pi - t) \text{ or } (2n\pi + t) \text{ for all } n \in \mathbb{N}\}$,

then the assertion of Theorem 1 remains valid.

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